UNIT 7 : LIMIT OF FUNCTIONS

UNIT STRUCTURE

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7.1 LEARNING OBJECTIVES

After going through this unit, you will able to-

- limit of a function
- algebra of limits
- right hand and left hand limits
- infinite limit.

7.2 INTRODUCTION

In this unit we will give a precise definition of limit due to the German mathematician Karl Thlep Theodor Wilhelm Weierstrass (1815-1897). Though Isaac Newton (1643-1727), d'Alembert (1717-1783), Cauchy (1789-1857) had a clear idea about limits, the rigorous definition known as $\varepsilon > \delta$ definition, stated by Weierstrass around 1850. We will also study algebra of limits, and related theorems, right hand limit and left hand limit, infinite limit.

7.3 LIMIT OF A FUNCTION

Let f be a function defined at all points near c, except possibly at c.

Let L be a real number

f is said to tend to the limit L as x tends to (or approaches to) c, if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

i.e. $|f(x) - L| < \varepsilon$, when $0 < |x - c| < \delta$

i.e. $f(x) \in]L - \varepsilon[, \forall x \in]c - \delta, c + \delta[$, except possibly at c.

Symbolically, we write $\lim_{x \to c} f(x) = L$

Nota Bene

1. We have $0 < |x - a| \Leftrightarrow x \neq a$. The inequality excludes the possibility x = a.
2. We take any $\varepsilon > 0$ and then choose some $\delta > 0$ i.e., The number ε is given first, and the number δ is to be chosen.
3. For each $\varepsilon > 0$, we can choose $\delta > 0$ such that if we choose x whose distance from c is less than δ, then the distance of f(x) from L must be less then ε.
4. The above definition does not give the value of L. It helps to examine whether a given number L is a limit or not.

Example 1 : Prove that $\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = 6$

Solution : We note that $x \neq 3$ (x approaches 3).

Now $\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} \frac{(x - 3)(x + 3)}{x - 3}$

$$= \lim_{x \to 3} (x + 3), \text{ as } x \neq 3$$

As x approaches 3, we can intuitively say that (x+3) approaches 6.

Now, we will prove that $\lim_{x \to 3} (x + 3) = 6$

Now f (x) – L = x+3 –6
= x – 3

which is in the form x – a, (∵ a = 3)

We take any $\varepsilon > 0$

Now $|f(x) - L| = |x + 3 - 6| < \varepsilon \Leftrightarrow |x - 3| < \varepsilon$

We choose $\delta > \varepsilon$

Thus, we see that

$$|x-3| < \delta = \varepsilon \Leftrightarrow |f(x)-L| = |x-3| < \varepsilon$$

$$\therefore \lim_{x \to 3} (x+3) = 6$$

Thus, $\lim_{x \to 3} \frac{x^2-9}{x-3} = 6$

Example 2. Prove that $\lim_{x \to 0} x \sin\frac{1}{x} = 0$

Solution : Let $\varepsilon > 0$ be given

We have

$$\left| x \sin\frac{1}{x} - 0 \right| = \left| x \sin\frac{1}{x} \right|$$

$$= |x| \left| x \sin\frac{1}{x} \right|$$

$$\leq |x|$$

$$\left(\because \left| \sin\frac{1}{x} \right| \leq 1 \right)$$

We choose $\delta = \varepsilon$, We get

$\left| x \sin\frac{1}{x} - 0 \right| < \varepsilon$, when $|x-0| < \delta$

$$\therefore \lim_{x \to 0} x \sin\frac{1}{x} = 0$$



CHECK YOUR PROGRESS 1

Q.1: Prove that $\lim_{x \to 0} \frac{x^2-4}{x-2} = 4$

Q.2: Prove that $\lim_{x \to 0} x^2 \sin\frac{1}{x} = 0$

Q.3: Prove that $\lim_{x \to 3} \frac{1}{x} = \frac{1}{3}, \; x \neq 0$

Theorem 7.1. Limit of a function, if exist, is unique.

Proof : Let $\lim_{x \to c} f(x)$ exist

If possile, let f(x) tend to two limits L_1 and L_2 as x tends to c.
$\therefore$ for any $\varepsilon > 0$, we can choose $\delta > 0$ such that

$|f(x) - L_1| < \frac{\varepsilon}{2}$, when $0 < |x - c| < \delta$

$|f(x) - L_2| < \frac{\varepsilon}{2}$, when $0 < |x - c| < \delta$

Now $|L_1 - L_2| = |L_1 - f(x) + f(x) - L_2|$

$\leq |L_1 - f(x)| + |f(x) - L_2|$

$< \varepsilon/2 + \varepsilon/2 = \varepsilon$, when $0 |x - c| < \delta$

i.e. $|L_1 - L_2|$ is less than any positive number ε, however small.

$\therefore L_1 - L_2$ must be zero, i.e. $L_1 - L_2$

Thus, the limit is unique.

Theorem 7.2. (Sandwich Theorem)

Let f, g, h be three functions defined at on $D \subseteq \underline{\underline{R}}$ containing c, except possibly at c.
Let

(i) $f(x) \leq g(x) \leq h(x), \forall x \in D - \{c\}$

(ii) $\lim_{x \to c} f(x) = L = \lim_{x \to c} h(x)$

Then $\lim_{x \to c} g(x)$ exists, and $\lim_{x \to c} g(x) = L$

Proof : We have $\lim_{x \to c} f(x) = L$

$\therefore$ given $\varepsilon > 0$, there exists $\delta_1 > 0$ such that

$|f(x) - L| < \varepsilon$, when $0 < |x - c| < \delta_1$

Agian, $\lim_{x \to c} h(x) = L$

$\therefore$ given $\varepsilon > 0$, there exists $\delta_2 > 0$ such that

$|h(x) - L| < \varepsilon$, when $0 < |x - c| < \delta_2$

Let $\delta = \min\{\delta_1, \delta_2\}$

$\therefore |f(x) - L| < \varepsilon$, when $0 < |x - c| < \delta$

i.e. $L - \varepsilon < f(x) < L + \varepsilon$, when $0 < |x - c| < \delta$

Also $|h(x) - L| < \varepsilon$, when $0 < |x - c| < \delta$

i.e. $L - \varepsilon < h(x) < L + \varepsilon$, when $0 < |x - c| < \delta$

We have $f(x) \le g(x) \le h(x), \forall x \in D - \{c\}$

$\therefore L - \varepsilon < f(x) \le g(x) \le h(x) < L + \varepsilon$, when $0 < |x - c| < \delta$

i.e. $L - \varepsilon < g(x) < L + \varepsilon$, when $0 < |x - c| < \delta$

i.e. $|g(x) - L| < \varepsilon$, when $0 < |x - c| < \delta$

i.e. $\lim_{x \to c} g(x) = L$

7.4 ALGEBRA OF LIMITS

Let f and g be two functions defined on $D \subseteq R$. We define f + g, f – g, fg, $\frac{f}{g}$ on D as follows

$$(f+g)(x) = f(x) + g(x), \quad \forall x \in D$$
$$(f-g)(x) = f(x) - g(x), \quad \forall x \in D$$
$$(fg)(x) = f(x)\, g(x), \quad \forall x \in D$$
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad \forall x \in D, \quad \text{provided} \quad g(x) \neq 0$$

Now we prove

Theorem 7.3. Let f, g be two functions defined at all points near c, except possibly at c. Let

$$\lim_{x \to c}(fx) = L, \quad \lim_{x \to c}(gx) = M$$

Then (i)

$$\lim_{x \to c}(f+g)(x) = \lim_{x \to c} f(x) - \lim_{x \to c} g(x) = L + M$$

(ii) $$\lim_{x \to c}(f-g)(x) = \lim_{x \to c} f(x) - \lim_{x \to c} g(x) = L - M$$

(iii) $\lim_{x \to c} (fg)\,(x) = \lim_{x \to c} f(x) \lim_{x \to c} g(x) = LM$

(iv) $\lim_{x \to c} \left(\frac{f}{g}\right)(x) = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \frac{L}{M}$, provided $M \neq 0$

Proof : (i) We have $\lim_{x \to c} f(x) = L$

$\therefore$ given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \frac{\varepsilon}{2}$,

when $0 < |x - c| < \delta$

Again $\lim_{x \to c} g(x) = M$

given $\varepsilon > 0$, there exists $\delta_2 > 0$

Such that

$|g(x) - M| < \frac{\varepsilon}{2}$, when $0 < |x - c| < \delta_2$

Let $\delta = \min\{\delta_1, \delta_2\}$

$\therefore |f(x) - L| < \frac{\varepsilon}{2}$, when $0 < |x - c| < \delta$

and $|g(x) - M| < \frac{\varepsilon}{2}$, when $0 < |x - c| < \delta$

Now $|(f + g)(x) - (L + M)|$

$= |f(x) - L + g(x) - M|$

$\leq |f(x) - L| + |g(x) - M|$

i.e. $|(f + g)(x) - (L + M)| < \varepsilon$, when $0 < |x - c| < \delta$

i.e. $\lim_{x \to c} (f + g) = L + M$

(ii) Proceeding as above, we have

$|(f - g)(x) - (L - M)|$

$|f(x) - g(x) - L + M|$

$$\left|\left(f(x)-L\right)+\left(M-g(x)\right)\right|$$

$$\leq \left|f(x)-L\right|+\left|M-g(x)\right|$$

$$< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon \text{, for } 0<\left|x-c\right|<\delta$$

i.e. $\left|(f-g)(x)-(L-M)\right|<\varepsilon$, for $0<\left|x-c\right|<\delta$

$$\lim_{x\to c}(f-g)=L-M$$

(iii) We have $\lim_{x\to c} f(x)=L$.

$\therefore$ given $\varepsilon_1>0$, there exist $\delta_1>0$ such that

$\left|f(x)-L\right|<\varepsilon_1$, when for $0<\left|x-c\right|<\delta$

Again, $\lim_{x\to c} g(x)=M$

$\therefore$ given $\varepsilon_2>0$, there exist $\delta_2>0$ such that

$\left|g(x)-M\right|<\varepsilon_2$, when for $0<\left|x-c\right|<\delta_2$

Let $\delta=\min\{\delta_1,\delta_2\}$

$\therefore \left|f(x)-L\right|<\varepsilon_1$,

and $\left|g(x)-M\right|<\varepsilon_2$, for $0<\left|x-c\right|<\delta$

Now $\left|g(x)\right|=\left|g(x)-M+M\right|$

$$\leq \left|g(x)-M\right|+\left|M\right|$$

$$< \varepsilon_2+\left|M\right|$$

$$< 1+\left|M\right|, \qquad \because \varepsilon_2<1$$

Now $\left|f(x)\,g(x)-LM\right|$

$$=\left|f(x)\,g(x)-L\,g(x)+L\,g(x)-LM\right|$$

$$=\left|g(x)\left(f(x)-L\right)+L\left(g(x)-M\right)\right|$$

$$=\left|g(x)\left(f(x)-L\right)\right|+\left|L\left(g(x)-M\right)\right|$$

$$=\left|g(x)\right|\left|f(x)-L\right|+\left|L\right|\left|g(x)-M\right|$$

$$<\left(1+\left|M\right|\right)\varepsilon_1+\left|L\right|\varepsilon_2 \text{, for } 0<\left|x-c\right|<\delta$$

We take $\varepsilon_1 = \dfrac{\varepsilon}{2(1+|M|)}$

and $\varepsilon_2 = \dfrac{\varepsilon}{2(1+|L|)}$

$\therefore$ (1) becomes

$|f(x)\ g(x) - LM|$

$< (1+|M|)\dfrac{\varepsilon}{2(1+|M|)} + |L|\dfrac{\varepsilon}{2(1+|L|)}$

$= \dfrac{\varepsilon}{2} + \dfrac{\varepsilon}{2}\cdot\dfrac{|L|}{1+|L|}$

$< \dfrac{\varepsilon}{2} + \dfrac{\varepsilon}{2}, \qquad \because \dfrac{|L|}{1+|L|} < 1$

$= \varepsilon$, for $0 < |x-c| < \delta$

i.e. $|f(x)\ g(x) - LM| < \varepsilon$, for $0 < |x-c| < \delta$

i.e. $\lim\limits_{x \to c} f(x)\ g(x) = LM$

(iv) Firstly, we prove that

$$\lim_{x \to c} \frac{1}{g(x)} = \frac{1}{M},\ M \neq 0$$

we have $\lim\limits_{x \to c} g(x) = M$

$\therefore$ given $\varepsilon > 0$, there exists $\delta > 0$ such that

$|g(x) - M| < \varepsilon$, for $0 < |x-c| < \delta$

Now $|M| = |M - g(x) + g(x)|$

$\leq |M - g(x)| + |g(x)|$

$\leq \varepsilon + |g(x)|$, for $0 < |x-c| < \delta$

i.e. $\varepsilon + |g(x)| > |M|$, for $0 < |x-c| < \delta$

i.e. $|g(x)| > |M| - \varepsilon$, for $0 < |x-c| < \delta$

We take $\varepsilon = \dfrac{|M|}{2}$

$$\therefore |g(x)| > |M| - \frac{|M|}{2}$$

i.e. $|g(x)| > \frac{|M|}{2}$, for $0 < |x - c| < \delta$

i.e. $\frac{1}{|g(x)|} < \frac{2}{|M|}$, for $0 < |x - c| < \delta$

Again, $\lim_{x \to c} g(x) = M$

$\therefore$ given $\varepsilon_1 = \frac{|M|^2}{2}\ \varepsilon > 0$, there exists $\delta > 0$ such that

$|g(x) - M| < \varepsilon_1$, for $0 < |x - c| < \delta$ (2)

$$\therefore \left| \frac{1}{g(x)} - \frac{1}{M} \right| = \left| \frac{M - g(x)}{M\, g(x)} \right|$$

$$= \frac{|g(x) - M|}{|M|\,|g(x)|}$$

$$< \frac{\varepsilon_1}{|M|} \cdot \frac{2}{|M|}, \text{ for } 0 < |x - c| < \delta$$

(from 1 and 2)

$$< \frac{2}{|M|^2} \cdot \frac{|M|^2}{2} \varepsilon, \text{ for } 0 < |x - c| < \delta$$

i.e. $\left| \frac{1}{g(x)} - \frac{1}{M} \right| < \varepsilon$, for $0 < |x - c| < \delta$

i.e. $\lim_{x \to c} \frac{1}{g(x)} = \frac{1}{M}$

Now $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} f(x) . \frac{1}{g(x)}$

$$= \left(\lim_{x \to c} f(x) \right) \left(\lim_{x \to c} \frac{1}{g(x)} \right)$$

$$= \frac{L}{M}$$

Do You Know ?

I. Let f(x) = k, for all x,where k is a constant; i.e. f is a constant function. Then $\lim_{x \to c} f(x) = k$

$|f(x) - L| = |k - k| = 0 < \varepsilon$, whatever may be the value δ

II. Let f(x) = x, for all x, i.e., f is an identity function. Then

$\lim_{x \to c} f(x) = c$.

$|f(x) - L| = |x - c| < \varepsilon$, for $0 < |x - c| < \delta$ choosing $\delta = \varepsilon$.



CHECK YOUR PROGRESS 2

Q 1: If $|f(x) - 2| \le 5(x+3)^2$ for all real x, find

$\lim_{x \to -3} f(x)$

Q 2: Prove that $\lim_{x \to 0} \frac{\text{Sin } x}{x} = 1$

Q 3: Prove that $\lim_{x \to 0} x \text{ Cos} \frac{1}{x} = 0$

Theorem 7.4. (Cauchy's Principle)

A necessary and sufficient condition for the existence of $\lim_{x \to c} f(x)$ is that for a preassigned number $\varepsilon > 0$, however small, there exists $\delta > 0$ such that x_1, x_2 being any two numbers satisfying for $0 < |x - c| < \delta$, $|f(x_1) - f(x_2)| < \varepsilon$

Proof : [We will only prove the necessary part].

Let $\lim_{x \to c} f(x) = L$

Then given $\varepsilon > 0$, there exists $\delta > 0$ such that

$|f(x) - L| < \varepsilon$, for $0 < |x - c| < \delta$

Let x_1, x_2 be any two numbers satisfying $0 < |x - c| < \delta$

Then $|f(x_1) - f(x_2)|$

$= |f(x_1) - L + L - f(x_2)|$

$$\leq |f(x_1) - L| + |L - f(x_2)|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

i.e. $|f(x_1) - f(x_2)| < \varepsilon$

Thus the condition is necessary.



CHECK YOUR PROGRESS 3

Q 1: Prove that $\lim_{x \to 0} \text{Cos}\frac{1}{x}$ does not exsit $(x \neq 0)$

Q 2: Prove that $\lim_{x \to 0} \frac{1}{2 + e^{\frac{1}{x}}}$ does not exist.

7.5 RIGHT-HAND AND LEFT-HAND LIMITS

In some cases, x approaches the number c form the right that is, through numbers bigger than c or from the left-that is, through numbers smaller than c. In some cases the function may be defined to the left or to the right of c. For example f(x) = /x is defined only for $x \geq 0$.

Definition 1. (Right-hand limit)

A function f is said to tend to a limit L as x approaches c from the right if for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$|f(x) - L| < \varepsilon$, when $c < x < c + \delta$

Symbolically, we write

$$\lim_{x \to c+0} f(x) = L \quad \text{or} \quad \lim_{x \to c+} f(x) = L$$

Definition 2 (Left-hand limit)

A function f is said to tend to a limit L as x approaches c from the left if for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$|f(x) - L| < \varepsilon$, when $c - \delta < x < c$

Symbolically, we write

$$\lim_{x \to c-0} f(x) = L \quad \text{or} \quad \lim_{x \to c-} f(x) = L$$

Do You Know ?

$0 < |x - c| < \delta$ if and only if either $c - \delta < x < c$ or $c < x < c + \delta$.

Theorem 7.4 Let f(x) be defined on an open interval containing c. Then

$\lim_{x \to c} f(x)$ is defined if and only if $\lim_{x \to c+} f(x)$ and $\lim_{x \to c-} f(x)$ are both defined and equal.

In this case

$$\lim_{x \to c} f(x) = \lim_{x \to c+} f(x) = \lim_{x \to c-} f(x)$$

Proof : Let $\lim_{x \to c} f(x)$ exist, and let $\lim_{x \to c} f(x) = L$

Then given $\varepsilon > 0$, there exists $\delta > 0$ such that

$|f(x) - L| < \varepsilon$, when $0 < |x - c| < \delta$.

Consequently $\lim_{x \to c+} f(x) = L$

Again, if $c - \delta < x < c$, then $0 < |x - c| < \delta$.

Consequently $\lim_{x \to c-} f(x) = L$

Conversely, let $\lim_{x \to c+} f(x) = L$

$\lim_{x \to c-} f(x) = L$

We have $\lim_{x \to c+} f(x) = L$

$\therefore$ given $\varepsilon > 0$, there exists $\delta_1 > 0$ such that

$|f(x) - L| < \varepsilon$, when $c < x < c + \delta_1$

Again, $\lim_{x \to c-} f(x) = L$

$\therefore$ given $\varepsilon > 0$, there exists $\delta_2 > 0$ such that

$|f(x) - L| < \varepsilon$, when $c - \delta_2 < x < c$

Let $\delta = \min\{\delta_1 . \delta_2\}$

Let $0 < |x - c| < \delta$

This implies either $c - \delta < x < c$ or $c < x < c + \delta$

Now $c - \delta_2 \le c - \delta < x < c$

$\therefore |f(x) - L| < \varepsilon$

Again $c < x < c + \delta \le c + \delta$

$\therefore |f(x) - L| < \varepsilon$

Thus $\lim_{x \to c} f(x) = L$



CHECK YOUR PROGRESS 4

Q 1: Evalute (i) $\lim_{x \to 0} \frac{|\sin x|}{\sin x}$

(ii) $\lim_{x \to 2} f(x)$, where

$$f(x) = \begin{cases} 8x + 7, \text{ if } x \ge 2 \\ x^2 + 19, \text{ if } x < 2 \end{cases}$$

Do You Know?

The greatest integer function (or the floor function), denoted [x], is defined as

[x] = the greatest integer less than or equal to x, $x \in R$

Thus [0.0005] = 0

[0.5] = 0, etc

It was first defined by the French mathematician Adrien-Marie Legendre (1752-1833] in 1798.

Example 3 Examine if $\lim_{x \to 0} [x]$ exists?

In $0 < x < 1$, $[x] = 0$

$\therefore \lim_{x \to 0+} [x] = 1$

In $-1 < x < 0$, $[x] = -1$

$$\therefore \lim_{x \to 0-} [x] = -1$$

Thus, $\lim_{x \to 0+} [x] \neq \lim_{x \to 0-} [x]$

$$\therefore \lim_{x \to 0} [x]$$

Do You Know?

The sign function or signum function, denoted by sgn x, is defined by

$$\text{sgn } x = \begin{cases} -1, \text{ if } x < 0 \\ 0, \text{ if } x = 0 \\ 1, \text{ if } x > 0 \end{cases}$$

Also $x = |x| \text{ sgn } x$,

$\text{sgn } x^n = (\text{sgn } x)^n$

$\frac{d|x|}{dx} = \text{sgn } x$, for $x \neq 0$

Example 4 Examine if $\lim_{x \to 0} [x]$ exists

When x > 0, sgn x =1

$$\therefore \lim_{x \to 0+} \text{sgn } x = 1$$

When x < 0, sgn x = –1

$$\therefore \lim_{x \to 0-} \text{sgn } x = -1$$

Thus $\lim_{x \to 0+} \text{sgn } x = 1 \neq \lim_{x \to 0-} \text{sgn } x = -1$

$$\therefore \lim_{x \to 0} \text{sgn } x$$ does not exist.

Example 5 Examine if $\lim_{x \to 0} f(x)$ exist, where f(x) = ex sgn (x + [x])

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} e^x \text{ sgn } (x + [x]) = -1$$

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} e^{x} \ \text{sgn}\,(x+[x]) = 1$$

$\therefore \lim_{x \to 0} f(x)$ does not exist.

Example 6 Examine if $\lim_{x \to a} f(x)$ exists, where

$$f(x) = \begin{cases} \dfrac{x^2}{a^2} - a, & \text{for } 0 < x < a \\ 0, & \text{for } x = a \\ a - \dfrac{a^2}{x^2}, & \text{for } x > a \end{cases}$$

$$\lim_{x \to a-} f(x) = \lim_{x \to a-} \left(\frac{x^2}{a^2} - a \right) = 0$$

$$\lim_{x \to a+} f(x) = \lim_{x \to a+} \left(a - \frac{a^2}{x^2} \right) = 0$$

$$\therefore \lim_{x \to a} f(x) = 0$$

7.6 INFINITE LIMITS

Let us consider the function $f(x) = \dfrac{1}{x}, \quad x > 0$

Clearly, it is a decreasing function. Also we note that f(x) comes closer and closer to zero as x becomes larger and larger. In this case we say that f(x) tends to zero as x tends to ∞.

Definition 1 A function f is said to tend to a limit L as x tends to ∞, if for every $\varepsilon > 0$, there is G > 0 such that $|f(x) - L| < \varepsilon$, when x > G symbolically, we write $\lim_{x \to \infty} f(x) = L$.

In source cases, we are to examine the behaviour of a function f(x), as x takes smaller and smaller negative values.

Definition 2 A function f is said to tend to a limit L as x tends to $-\infty$, if for

every $\varepsilon > 0$, there is G > 0 such that $|f(x) - L| < \varepsilon$, when x < –G.

Symbolically, we write $\lim_{x \to \infty} f(x) = L$

Example 7 Prove that $\lim_{x \to \infty} \frac{1}{x} = 0$,

Here $f(x) = \frac{1}{x}, \quad x \neq 0$

We take $\varepsilon > 0$

Now $|f(x) - L| = \frac{1}{x} - 0 = \left|\frac{1}{x}\right|, \ (x \neq 0)$

$\therefore |f(x) - 0| < \varepsilon$, whenever x > G, where $G = \frac{1}{\varepsilon} > 0$

$\therefore \lim_{x \to \infty} f(x) = 0$

Example 8 Prove that $\lim_{x \to \infty} \left(\frac{3}{x} + \frac{5}{x^2} + 7\right) = 0$

We have $\lim_{x \to \infty} \frac{1}{x} = 0$

Now we prove that $\lim_{x \to \infty} \frac{1}{x^2} = 0$

We take $\varepsilon > 0$

Now $\left|\frac{1}{x^2} - 0\right| = \frac{1}{x^2}, \ (x \neq 0)$

$\therefore \left|\frac{1}{x^2} - 0\right| < \varepsilon$, whenever x > G, where $G = \frac{1}{\sqrt{\varepsilon}} > 0$

$\therefore \lim_{x \to \infty} \frac{1}{x^2} = 0$

Now, $\lim_{x \to \infty} \left(\frac{3}{x} + \frac{5}{x^2} + 7\right)$

$= 3 \lim_{x \to \infty} \frac{1}{x} + 5 \lim_{x \to \infty} \frac{1}{x^2} + \lim_{x \to \infty} 7 = 0 + 0 + 7 = 7$



CHECK YOUR PROGRESS 5

Q 1: Prove that $\lim_{x \to \infty} \frac{x^2}{1+x^2} = 1$

Q 2: If –1 < x < 1, prove that $\lim_{x \to \infty} x^n = 0$ (n is a positive integer)



7.7 LET US SUM UP

- Let f be a function defined at all points near c, except possibly at c.
  Let L be a real number.
  f is said to tend to the limit L as x tends to c, if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that
  $0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$
- The $\varepsilon - \delta$ definition does not give the value of L. It helps to examine whether a given number L is a limit or not.
- Limit of a function, if exists, is unique.
- Let f, g, h be three functions defined on $D \subseteq R$ containing c, except possibly at c.
  Let
  (i) $f(x) \le g(x) \le h(x), \quad \forall x \in D - \{c\}$
  (ii) $\lim_{x \to c} f(x) = L = \lim_{x \to c} h(x)$

  then, $\lim_{x \to c} g(x)$ exists, and $\lim_{x \to c} g(x) = L$

  This theorem is known as Sandwich Theorem.
- A function f is said to tend to a limit L as x tends to ∞, if for every $\varepsilon > 0$, there is G > 0 such that
  $|f(x) - L| < \varepsilon$, when x > G
  Symbolically, we write $\lim_{x \to \infty} f(x) = L$

7.8 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress 1

Ans to Q No 1: We note that $x \neq 2$ (x approaches 2)

Now $\lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} \frac{(x-2)\,(x+2)}{x-2}$

$$= \lim_{x \to 2} (x+2), \quad \text{as } x \neq 2$$

As x approaches 2, we can intnitively say that (x+2) approaches 4.

Now, we will proe that $\lim_{x \to 2} (x+2) = 4$

Now f (x) – L = x + 2 – 4 = x – 2, which is in the form x – a, (a = 2)

We take any $\varepsilon > 0$

Now $|f(x)-L| = |x+2-4| < \varepsilon \Leftrightarrow |x-2| < \varepsilon$

We choose $\delta = \varepsilon$

Thus, we see that

$|x-2| < \delta = \varepsilon \Rightarrow |f(x)-L| = |x-2| < \varepsilon$

$\therefore \lim_{x \to 2} (x+2) = 4$

Thus, $\lim_{x \to 2} \frac{x^2-4}{x-2} = 4$

Ans to Q No 2: Here $f(x) = x^2 \ \text{Sin} \frac{1}{x}$

$$|f(x)-0| = \left|x^2 \text{Sin}\frac{1}{x}\right| = |x^2| \left|\text{Sin}\frac{1}{x}\right|$$

$$\leq |x^2| \qquad \left(\because \left|\text{Sin}\frac{1}{x}\right| \leq 1\right)$$

$\therefore |f(x)-0| < \varepsilon$, when $0 < |x^2| < \varepsilon$

i.e. $|f(x)-0| < \varepsilon$, when $0 < |x| < \varepsilon^{\frac{1}{2}}$

We choose $\delta = \varepsilon^{\frac{1}{2}}$

$\therefore |f(x)-0| < \varepsilon$, when $0 < |x-0| < \delta$

i.e. $\lim_{x \to 0} x^2 \operatorname{Sin}\frac{1}{x} = 0$

Ans to Q No 3: Here $f(x) = \frac{1}{x}$

$$\left|f(x)-\frac{1}{3}\right| = \left|\frac{1}{x}-\frac{1}{3}\right| = \left|\frac{3-x}{3x}\right| = \frac{|x-3|}{3|x|}$$

We choose $\delta \le 1$, and $0 < |x-3| < \delta$ then $0 < |x-3| < 1$

$\Rightarrow x \neq 3$ and $3-1 < x < 3+1$

$\Rightarrow x \neq 3$ and $2 < x < 4$

$\Rightarrow x \neq 3$ and $\frac{1}{2} > \frac{1}{x} > \frac{1}{4}$

$\Rightarrow x \neq 3$ and $\frac{1}{4} < \frac{1}{x} < \frac{1}{2}$

$\Rightarrow x \neq 3$ and $\frac{1}{|x|} < \frac{1}{2}$, $\left[\because \frac{1}{x} > \frac{1}{4} > 0,\ \therefore \frac{1}{x} = \frac{1}{|x|}\right]$

$$\therefore \left|f(x)-\frac{1}{3}\right| = \frac{|x-3|}{3}.\frac{1}{|x|} < \frac{\delta}{3}.\frac{1}{2} = \frac{8}{6}$$

Now, we choose δ such that $\frac{\delta}{6} < \varepsilon$, i.e. $\delta < 6\varepsilon$

Also $\delta \le 1$

$\therefore$ taking $\delta = \min\{1, 6\varepsilon\}$, we have

$\left|f(x)-\frac{1}{3}\right| < \frac{\delta}{6} < \varepsilon$, whenever $0 < |x-3| < \delta$

$\therefore \left|f(x)-\frac{1}{3}\right| < \varepsilon$, whenever $0 < |x-3| < \delta$

$\therefore \lim_{x \to 3} f(x) = \frac{1}{3}$

Check Your Progress 2

Ans to Q No 1: $|f(x)-2| \le 5(x+3)^2$ (given)

$$\Rightarrow 2-5\,(x+3)^2 \le f(x) \le 2+5\,(x+3)^2$$

Now $\lim_{x \to 3} \left(2-5(x+3)^2\right) = 2$

$$\lim_{x \to 3} \left(2+5(x+3)^2\right) = 2$$

$\therefore$ by Sandwich Theorem, $\lim_{x \to 3} f(x) = 2$

Ans to Q No 2: Here $f(x) = \frac{\sin x}{x},\ x \neq 0$

In $0 < x < \frac{\pi}{2},\ \sin x < x < \tan x$

$$\therefore\ 1 < \frac{x}{\sin x} < \frac{1}{\cos x}$$

$\therefore\ \lim_{x \to 0+} 1 = 1$ and $\lim_{x \to 0+} \frac{1}{\cos x} = 1$

$\therefore$ by Sandwich Theorem, $\lim_{x \to 0+} \frac{x}{\sin x} = 1$

i.e. $\lim_{x \to 0+} \frac{x}{\sin x} = 1$

Again, in $-\frac{\pi}{2} < x < 0,\ \tan x < x < \sin x$

$$\therefore 1 < \frac{x}{\sin x} < \frac{1}{\cos x}$$

Now $\lim_{x \to 0-} 1 = 1$, and $\lim_{x \to 0-} \frac{1}{\cos x} = 1$

$\therefore$ by Sandwich Theorem, $\lim_{x \to 0-} \frac{1}{\sin x} = 1$

i.e. $\lim_{x \to 0-} \frac{\sin x}{x} = 1$

Thus, $\lim_{x \to 0+} \frac{\sin x}{x} = 1 = \lim_{x \to 0-} \frac{\sin x}{x}$

$$\therefore \lim_{x \to 0} \frac{\text{Sin } x}{x} = 1$$

Ans to Q No 3: Try Yourself

Check Your Progress 3

Ans to Q No 1: Try yourself

Ans to Q No 2: Try yourself

Check Your Progress 4

Ans to Q No 1: (i) We have

$$\lim_{x \to 0+} \frac{|\text{Sin } x|}{\text{Sin } x} = 1, \quad \text{whereas} \lim_{x \to 0-} \frac{|\text{Sin } x|}{\text{Sin } x} = -1$$

$$\therefore \lim_{x \to 0} \frac{|\text{Sin } x|}{\text{Sin } x} \text{ does not exist}$$

(ii) Try yourself

Check Your Progress 5

Ans to Q No 1: We have

$$\left| \frac{x^2}{1+x^2} - 1 \right| = \left| \frac{x^2 + 1 - x^2}{1+x^2} \right| = \left| \frac{1}{1+x^2} \right| = \frac{1}{1+x^2}$$

Let $\varepsilon > 0$

Now $\frac{x^2}{1+x^2} < \varepsilon$, whenever $1 + x^2 > \frac{1}{\varepsilon}$

i.e., whenever $x^2 > \frac{1}{\varepsilon} - 1$

i.e. whenever $x > \left| \frac{1}{\varepsilon} - 1 \right|^{\frac{1}{2}}$

i.e. $\frac{x^2}{1+x^2} < \varepsilon$, whenever x > G, where $G = \left| \frac{1}{\varepsilon} - 1 \right|^{\frac{1}{2}}$

i.e. $\left| \frac{x^2}{1+x^2} - 1 \right| < \varepsilon$, whenever x > G

$$\therefore \lim_{x \to \infty} \frac{x^2}{1+x^2} = 1$$



7.9 FURTHER READING

1) Fraleigh, John B : A first course in abstract Algebra, Pearson Education (Singapure) Pvt. Ltd. India Branch, Delhi - 11092.

2) Singh, S and Qazi, Z : Modern Algebra, Vikas Publicasting House Pvt. Ltd, New Delhi - 110014.

3) Macauley, M. : Section 6 : Fields and Galois Theory, http :// www.math.clemson.edu/~macaule/.

4) Talati, A. : Class Notes of Professor June Huh, 2022.

5) Singh, J. : Theory of Field Extension, Maharshi Dayanand University Press, Rohtal-124001.



7.10 MODEL QUESTIONS

Q 1: It is given that

Express it in $\varepsilon-\delta$ form

[There exists $\varepsilon>0$, such that, for all $\delta>0$, there exists x such that $|x-c|<\delta$ and $|f(x)-L|>\varepsilon$].

Q 2: Evaluate $\lim\limits_{x\to 2} f(x)$, if exists, where

$$f(x)=\begin{cases}\dfrac{|x-2|}{x-2}, & x\neq 2\\ 2, & x\neq 2\end{cases}$$

Q 3: If $\lim\limits_{x\to c} f(x)=L$, show that $\lim\limits_{x\to c}|f(x)|=|L|$

Q 4: Show that $\lim\limits_{x\to 0}\dfrac{e^{\frac{1}{x}}}{e^{\frac{1}{x}+1}}$ does not exist.

Q 5: Evaluate $\lim_{x \to 0+} \frac{\text{Sin } x}{\sqrt{x}}$

Q 6: Prove that $\lim_{x \to 0} \sqrt{x}\ \text{Sin}\frac{1}{x} = 0$

Q.7: Show that $\lim_{x \to 0} \text{Sin}\frac{1}{x}$ does not exist.

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